STANZALEAF INTERNATIONAL JOURNAL OF MULTIDISCIPLINARY STUDIES (SLIJMS)
(ISSN:* 3139-3373*)
Volume 2 Issue 2 June 2026 - Available at https://slijms.com

On the Analytical Relationship between the Fractional Hilbert Transform and Fourier Transform

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Abstract

The fractional Hilbert transform is a generalization of the classical Hilbert transform and plays an important role in various branches of engineering and science. In this paper, new analytical relationships between the fractional Hilbert transform and the classical Fourier transform are established. The results are derived using the definition of the fractional Hilbert transform and properties of the Fourier transform in the generalized function framework. Several fundamental transform identities are obtained, and the corresponding classical Hilbert transform relations are recovered as special cases. The derived theorems provide a useful theoretical framework for the analysis of fractional integral transforms and may find applications in solving differential equations, signal processing, and related areas of applied mathematics.

Keywords: *Fourier transform, fractional Fourier transform, Fractional Hilbert transform.*

1. INTRODUCTION

The Hilbert transform based on Fourier transform has widely been applied in many areas such as optical systems, modulation, edge detections [1, 3], etc. The Fourier transform is a most popular transform in the theory of optics, signal processing and many other branches of engineering. Namias [4] introduced the concept of Fourier transform of fractional order by the method of eigen values. Zayed [5] had generalized the definition of the fractional Hilbert transform in order to obtain the analytic signals that is associated with its fractional Fourier transform.

The novelty of this work lies in establishing explicit analytical relationships between the fractional Hilbert transform and the classical Fourier transform using generalized function theory. The derived identities extend classical transform relations and provide a framework for further applications.

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The paper is organized as follows. Section 2 presents the definitions and preliminary concepts. Section 3 establishes the main analytical relationships between the fractional Hilbert transform and the Fourier transform. Section 4 concludes the paper.

2 DEFINITIONS

2.1 The fractional Hilbert transform (FrHT) of function $f(x)$ defined in [5] is

$$H^{\alpha}[f(x)](t) = \tilde{f}(t)$$

$$= \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x}\, e^{i\frac{\cot\alpha}{2}x^2}\, dx \text{ , for } \quad t \in \mathbb{R}, t \neq x \text{ }, \alpha \neq 0, \frac{\pi}{2}, \pi$$

where the integral is evaluated in the Cauchy principal value sense.

2.2 The Fourier transform of a function $f(t)$ defined in [2] is

$$F\left[f(t)\right](u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iut} dt$$

3. RELATION BETWEEN THE FRACTIONAL HILBERT TRANSFORM AND THE FOURIER TRANSFORM

Unlike previous studies, this paper derives explicit analytical relationships between the fractional Hilbert transform and the Fourier transform in generalized function spaces.

In this section we defined $\overline{f}(x) = f(x)e^{i\frac{\cot\alpha}{2}x^2}$ and $\overline{\overline{f}}(x) = f(x)\, e^{-i\frac{\cot\alpha}{2}x^2}$ the required notation and establish the following analytical relationships.

Theorem 3.1 $H^{\alpha}[f(x)](t) = -\sqrt{\frac{2}{\pi}}\, F\left[\frac{f(t-p)}{p}\, e^{i\frac{\cot\alpha}{2}p^2}\right](t\cot\alpha)$

Proof : $H^{\alpha}[f(x)](t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x}\, e^{i\frac{\cot\alpha}{2}x^2} dx$

putting $t - x = \text{p}$ so that $-dx = dp$

$$= \frac{-e^{-i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{f(t-p)}{p}\, e^{i\frac{\cot\alpha}{2}(t-p)^2} dp$$

$$H^{\alpha}[f(x)](t) = -\sqrt{\frac{2}{\pi}} F\left[\frac{f(t-p)}{p} e^{i\frac{\cot\alpha}{2}p^2}\right](t\cot\alpha)$$

Theorem 3.2 $F\left\{e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[f(x)](t)\right\}(u) = -i\,\mathrm{sgn}(u) F\left[\overline{f}(x)\right](u)$

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Proof: $F\left\{e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[f(x)](t)\right\}(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[f(x)](t)\, e^{-iut}dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\frac{\cot\alpha}{2}t^2-iut}\left\{\frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx\right\}dt$$

By changing the order of integration (assuming the conditions of Fubini's theorem are satisfied)

$$= \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\frac{\cot\alpha}{2}x^2}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-iut}}{x-t}dt\right\}dx$$

$$= \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\frac{\cot\alpha}{2}x^2}\, i\,\mathrm{sgn}\,(u)\, e^{-iux}\, dx$$

$$= -i\,\mathrm{sgn}\,(u)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\frac{\cot\alpha}{2}x^2}\, e^{-iux}\, dx$$

$$F\left\{e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[f(x)](t)\right\}(u) = -i\,\mathrm{sgn}\,(u)\, F\left[\overline{f}(x)\right](u)$$

Theorem 3.3 $F\left\{e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[\overline{\overline{f}}(x)](t)\right\}(u) = -i\,\mathrm{sgn}(u)F[f(x)](u)$

Proof: Let $g(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\overline{\overline{f}}(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$

$$e^{-i\frac{\cot\alpha}{2}t^2} g(t) = \frac{e^{-i\frac{\cot\alpha}{2}t^2}}{\pi}\int_{-\infty}^{\infty}\frac{\overline{\overline{f}}(x)}{t-x}e^{i\frac{\cot\alpha}{2}x^2}dx$$

$$= H^{\alpha}[\overline{\overline{f}}(x)](t)$$

$$F\left\{e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[\overline{\overline{f}}(x)](t)\right\}(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[\overline{\overline{f}}(x)](t)e^{-iut}dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iut}e^{-i\frac{\cot\alpha}{2}t^2} g(t)e^{i\frac{\cot\alpha}{2}t^2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iut}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\overline{\overline{f}}(x)}{t-x}\, e^{i\frac{\cot\alpha}{2}x^2}dx\right\}dt$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iut} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x} dx \right\} dt$$

By changing the order of integration

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-iut}}{x-t} dt \right\} dx$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) i \operatorname{sgn}(u) e^{-iux} dx$$

$$= -i \operatorname{sgn}(u) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iux} dx$$

$$F\left\{ e^{i\frac{\cot\alpha}{2}t^2} H^{\alpha}[\overline{\overline{f}}(x)](t) \right\}(u) = -i \operatorname{sgn}(u) F[f(x)](u)$$

Theorem 3.4 $F\left\{ e^{-i\frac{\cot\alpha}{2}t^2} H^{-\alpha}[f(x)](t) \right\}(u) = i \operatorname{sgn}(u) F[\overline{\overline{f}}(x)](u)$

Proof: Let $g(t) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x} e^{-i\frac{\cot\alpha}{2}x^2} dx$

$$e^{i\frac{\cot\alpha}{2}t^2} g(t) = \frac{-e^{i\frac{\cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x} e^{-i\frac{\cot\alpha}{2}x^2} dx$$

$$= H^{-\alpha}[f(x)](t)$$

$$F\left\{ e^{-i\frac{\cot\alpha}{2}t^2} H^{-\alpha}[f(x)](t) \right\}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\frac{\cot\alpha}{2}t^2} H^{-\alpha}[f(x)](t) e^{-iut} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iut} e^{-i\frac{\cot\alpha}{2}t^2} g(t) e^{i\frac{\cot\alpha}{2}t^2} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iut} \left\{ \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x} e^{-i\frac{\cot\alpha}{2}x^2} dx \right\} dt$$

By changing the order of integration

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\frac{\cot\alpha}{2}x^2} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-iut}}{x-t} dt \right\} dx$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\frac{\cot\alpha}{2}x^2} \, i \operatorname{sgn}(u) e^{-iux} \, dx$$

$$F\left\{ e^{-i\frac{\cot\alpha}{2}t^2} H^{-\alpha}[f(x)](t) \right\}(u) = i \operatorname{sgn}(u) F[\overline{\overline{f}}(x)](u)$$

Theorem 3.5

$$H^{\alpha}[f(x)](t) = \frac{-1}{\sqrt{2\pi}} \left\{ F[\frac{f(t+p)}{p} e^{i\frac{\cot\alpha}{2}p^2}](-t\cot\alpha) + F\left[\frac{f(t-p)}{p} e^{i\frac{\cot\alpha}{2}p^2}\right](t\cot\alpha) \right\}$$

Proof: Using Theorem 3.1 and 3.2

$$2H^{\alpha}[f(x)](t) = -\sqrt{\frac{2}{\pi}} \left\{ F[\frac{f(t+p)}{p} e^{i\frac{\cot\alpha}{2}p^2}](-t\cot\alpha) + F\left[\frac{f(t-p)}{p} e^{i\frac{\cot\alpha}{2}p^2}\right](t\cot\alpha) \right\}$$

$$H^{\alpha}[f(x)](t) = \frac{-1}{\sqrt{2\pi}} \left\{ F[\frac{f(t+p)}{p} e^{i\frac{\cot\alpha}{2}p^2}](-t\cot\alpha) + F\left[\frac{f(t-p)}{p} e^{i\frac{\cot\alpha}{2}p^2}\right](t\cot\alpha) \right\}$$

Note that for $\alpha = \frac{\pi}{2}$, we get relations between classical Hilbert transform and Fourier transform [2].

4. CONCLUSION

In this paper, new analytical relationships between the fractional Hilbert transform and the classical Fourier transform have been established. The derived theorems provide explicit transform-domain representations that extend the corresponding classical Hilbert transform identities. Furthermore, the classical relationships are recovered as special cases of the obtained results. Future work may focus on extending these relationships to other fractional integral transforms and investigating their applications in solving practical engineering and mathematical problems.

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